

# PRIME COUNTING FUNCTION

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### ABSTRACT

The theorem below gives another way of computing the distribution prime counting function without using recursion and the values of Prime numbers

### THEOREM

The prime counting function is the function  $\pi(n)$ , giving the number of primes less than or equal to a given number  $n$ , in explicit form is expressed by the formula:

$$\pi(n) = n - 1 - \sum_{i=2}^{[\sqrt{n}]} \left( \left[ \frac{n}{i} \right] - i + 1 \right) + \sum_{s=2}^{[\sqrt{n}]} (-1)^s \sum_{1 < i_1 < i_2 < \dots < i_s \leq [\sqrt{n}]} \left( \left[ \frac{n}{LCM(i_1, i_2, \dots, i_s)} \right] - \left[ \frac{i_s^2 - 1}{LCM(i_1, i_2, \dots, i_s)} \right] \right)$$

Where the  $[x]$  is the floor function of  $x$ ,  $LCM(i_1, i_2, \dots, i_s)$  the least common multiple of positive integers  $i_1, i_2, \dots, i_s$ .

### PROOF:

Function  $\pi(n)$  is equal to a difference  $n - 1$  (1 - by definition not prime) and numbers of compound numbers, each of which is  $i \cdot j$  with conditions:  $i \cdot j \leq n$ ,  $i, j \in \mathbb{N}_1$ , where through  $\mathbb{N}_1$  we shall designate set of natural numbers, greater 1.

Let's add a condition  $j \geq i$  which does not limit a generality.

Thus, all natural compound numbers, smaller or equal  $n$ , form set:

$$X = \{x \mid x \leq n, x = i \cdot j, j \geq i, x, i, j \in \mathbb{N}_1\}.$$

By definition  $\pi(n) = n - 1 - |X|$ .

Let's designate through  $X_i$  set of natural compound numbers of a type  $i \cdot j$ , not surpassing  $n$ , with fixed  $i \in N_1$ :

$$X_i = \{x \mid x \leq n, x = k \cdot j, k = i, j \geq k, x, j, k \in \mathbb{N}_1\}.$$

Note that  $X_i = \emptyset$ ,  $\forall i > [\sqrt{n}]$ .

For clarity, draw a table, for example, for  $n = 11$ :

$$\begin{array}{cccccccccccccccc}
 j & & & & & & & & & & & & & & \\
 11 & 11 & & & & & & & & & & & & & \\
 10 & 10 & & & & & & & & & & & & & \\
 9 & 9 & & & & & & & & & & & & & \\
 8 & 8 & & & & & & & & & & & & & \\
 7 & 7 & & & & & & & & & & & & & \\
 6 & 6 & & & & & & & & & & & & & \\
 5 & 5 & 10 & & & & & & & & & & & & \\
 4 & 4 & 8 & & & & & & & & & & & & \\
 3 & 3 & 6 & \mathbf{9} & & & & & & & & & & & \\
 2 & 2 & 4 & 6 & 8 & 10 & & & & & & & & & \\
 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & i & & 
 \end{array}$$

In an example  $X_2 = \{4, 6, 8, 10\}$ ,  $X_3 = \{9\}$ .

In the table  $X_k$  is all numbers from  $k$ -th column, it is more or equal  $k^2$ . It's clear that:

$$X = X_2 \cup X_3 \cup X_4 \cup \cdots \cup X_{\lfloor \sqrt{n} \rfloor}.$$

In columns numbers  $i_1$  and  $i_2$  there can be identical numbers, i.e.  $X_{i_1} \cap X_{i_2}$  is not always empty.

In accordance with the Inclusion-Exclusion Principle [1]:

$$|X| = \sum_{i=2}^{\lfloor \sqrt{n} \rfloor} |X_i| - \sum_{s=2}^{\lfloor \sqrt{n} \rfloor} (-1)^s \sum_{1 \leq i_1 < i_2 < \dots < i_s \leq \lfloor \sqrt{n} \rfloor} |X_{i_1} \cap X_{i_2} \cap \dots \cap X_{i_s}|$$

Finding power sets, the right-hand side of this equation, we will find  $\pi(n)$ .

Let us prove the following statement.

## STATEMENT

For  $\forall i, s, i_1, i_2, \dots, i_s \in N_1, 1 < i_1 < i_2 < \dots < i_s \leq [\sqrt{n}]$ ,  
 $X_i = \{x \mid x \leq n, x = k \cdot j, k = i, j \geq k, x, j, k \in N_1\}$  right:

$$|X_{i_1} \cap X_{i_2} \cap \dots \cap X_{i_s}| = \left\lfloor \frac{n}{LCM(i_1, i_2, \dots, i_s)} \right\rfloor - \left\lfloor \frac{i_s^2 - 1}{LCM(i_1, i_2, \dots, i_s)} \right\rfloor,$$

Where the  $[x]$  is the floor function of  $x$ ,  $LCM(i_1, i_2, \dots, i_s)$  the least common multiple of positive integers  $i_1, i_2, \dots, i_s$ .

## PROOF:

Let  $Y_i(n)$  the set of natural composite numbers of the form  $i \cdot j$ , not exceeding  $n$ , with a fixed  $i \in N_1$ , without the condition  $j \geq i, j \in N$ :

$$Y_i(n) = \{y \mid y \leq n, y = k \cdot j, k = i, y, k \in N_1, j \in N\}.$$

$Y_k(n)$  - set of numbers, standing in the  $k$ -th column of the table in question.  
 In an example:  $Y_2(11) = \{2, 4, 6, 8, 10\}$ ,  $Y_3(11) = \{3, 6, 9\}$ ,  $Y_2(3) = \{2\}$ ,  $Y_3(8) = \{3, 6\}$ .

From the above definition that sets  $\forall s, n, i_1, i_2, \dots, i_s \in N_1, 1 < i_1 < i_2 < \dots < i_s \leq [\sqrt{n}]$  true equality:

$$|X_{i_1} \cap X_{i_2} \cap \dots \cap X_{i_s}| = |Y_{i_1}(n) \cap Y_{i_2}(n) \cap \dots \cap Y_{i_s}(n)| - |Y_{i_1}(i_s^2 - 1) \cap Y_{i_2}(i_s^2 - 1) \cap \dots \cap Y_{i_s}(i_s^2 - 1)|$$

In the left part of equality there is a quantity of numbers in a column with the maximal index  $i_s$ , conterminous with numbers from columns with numbers  $i_1, i_2, \dots, i_{s-1}$ , a type  $i_s \cdot j$ , not surpassing  $n$ , with a condition  $j \geq i$ .

In the right part – a difference of quantity of the same numbers in the same  $i_s$ -th column without a condition  $j \geq i_s$  and quantities of the same numbers which size less or is equal  $i_s^2 - 1$ , that from a way of construction of the table to equivalently condition  $j < i_s$ .

We prove by induction on the index of  $s$ , that  $\forall s, n, i_1, i_2, \dots, i_s \in N_1$ :

$$Y_{i_1}(n) \cap Y_{i_2}(n) \cap \dots \cap Y_{i_s}(n) =$$

$$\{y \mid y \leq n, \exists m \in N : y = LCM(i_1, i_2, \dots, i_s) \cdot m, y \in N_1\}$$

and then:

$$|Y_{i_1}(n) \cap Y_{i_2}(n) \cap \dots \cap Y_{i_s}(n)| = \left\lfloor \frac{n}{LCM(i_1, i_2, \dots, i_s)} \right\rfloor.$$

For  $s = 1$ :

For  $\forall i \in N_1$

$$Y_i(n) = \{y \mid y \leq n, y = k \cdot j, k = i, y, k \in N_1, j \in N\},$$

which is equivalent to:

$$Y_i(n) = \{y \mid y \leq n, \exists j \in N : y = LCM(i) \cdot j, y \in N_1\}$$

and

$$|Y_i(n)| = \left\lfloor \frac{n}{i} \right\rfloor = \left\lfloor \frac{n}{LCM(i)} \right\rfloor.$$

The first step of the induction is confirmed by these equations.

Suppose that for  $s = k - 1$  is true equality, we show that for  $s = k$  is also true.

$$Y_{i_1}(n) \cap Y_{i_2}(n) \cap \dots \cap Y_{i_{k-1}}(n) = \{y \mid y \leq n, \exists m \in N : y = LCM(i_1, i_2, \dots, i_{k-1}) \cdot m, y \in N_1\},$$

$$\begin{aligned} & Y_{i_1}(n) \cap Y_{i_2}(n) \cap \dots \cap Y_{i_k}(n) = \\ & \{y \mid y \leq n, y = i_1 \cdot j_1 = i_2 \cdot j_2 = \dots = i_k \cdot j_k, y \in N_1, j_1, j_2, \dots, j_k \in N_1\} = \\ & \{y \mid y \leq n, \exists m, r \in N : y = LCM(i_1, i_2, \dots, i_{k-1}) \cdot m = i_k \cdot r, y \in N_1\} = \\ & \{y \mid y \leq n, \exists m \in N : y = LCM(LCM(i_1, i_2, \dots, i_{k-1}), i_k) \cdot m, y \in N_1\}, \end{aligned}$$

i.e.

$$|Y_{i_1}(n) \cap Y_{i_2}(n) \cap \dots \cap Y_{i_k}(n)| = \left\lfloor \frac{n}{LCM(LCM(i_1, i_2, \dots, i_{k-1}), i_k)} \right\rfloor = \left\lfloor \frac{n}{LCM(i_1, i_2, \dots, i_k)} \right\rfloor.$$

Therefore, by induction to  $\forall s \in N_1$ :

$$|Y_{i_1}(n) \cap Y_{i_2}(n) \cap \dots \cap Y_{i_s}(n)| = \left\lfloor \frac{n}{LCM(i_1, i_2, \dots, i_s)} \right\rfloor.$$

As a result,  $|X_{i_1} \cap X_{i_2} \cap \dots \cap X_{i_s}| = \left\lfloor \frac{n}{LCM(i_1, i_2, \dots, i_s)} \right\rfloor - \left\lfloor \frac{i_s^2 - 1}{LCM(i_1, i_2, \dots, i_s)} \right\rfloor$ , as proves the Statement.

Having substituted expression for capacity of intersection of sets  $X_{i_1}, X_{i_2}, \dots, X_{i_s}$  in the formula of inclusion-exclusion, we shall receive the proof of the basic theorem, in view of that in the first sum for presentation  $|X_i| = \left\lfloor \frac{n}{LCM(i)} \right\rfloor - \left\lfloor \frac{i^2 - 1}{LCM(i)} \right\rfloor = \left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{i^2 - 1}{i} \right\rfloor = \left\lfloor \frac{n}{i} \right\rfloor - \left[ i - \frac{1}{i} \right] = \left\lfloor \frac{n}{i} \right\rfloor - i + 1$ .

## REFERENCES:

- [1] Andrews, G. E. Number Theory. Philadelphia, PA: Saunders, pp. 139-140, 1971.